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Dynamics of Discrete-Space Structure

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We introduce a model for self-generating discrete-space structure based on the "local" quantum mechanics of graphs. In this approach, the dimension of space becomes a scale-dependent quantity (ρ dimension); it can, at least in principle, be calculated from the theory. We show that the straightforward use of the creation and annihilation operator approach (with standard commutation relations) leads to contradiction when applied to graphs. We define a reformulated Fock-space method based on a topological description. The properties of the theory are discussed in the framework of a simple solvable model.

1. INTRODUCTION

During the 50 years since the discovery of infinities (Pauli, 1931) in quantum relativistic field theories, various criticisms and alternatives to different basic axioms have been suggested. In most of these theories, a fundamental length (Heisenberg, 1930, 1938; Ambarzumian and Ivanenko, 1930; March, 1936, 1937; White, 1950, 1954) l_0 was introduced in such a way that a large-distance space–time structure remained unchanged, while for distances of order l_0, the structure was so modified that ultraviolet divergences did not appear.

The most straightforward way to formulate the above ideas was to use a cubic or a hypercubic lattice (Silberstein, 1963; Schild, 1948, 1949; Coxeter and Whitrov, 1950; Helund and Tamaka, 1954; Rojansky, 1955; Hill, 1955; Das, 1960; Meesen, 1967, 1972) as discrete space or space-time. In the gauge-fields-on-lattice approach (Wilson, 1974; Balian et al., 1974) to quark confinement, the use of the lattice is mainly technical since it enables one to perform straightforward calculations of higher-order terms

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in perturbation expansion. The common drawback of lattice theories is the lack of Lorentz or even rotational invariance. It is suggested that symmetry might be restored in the zero lattice spacing limit, or alternatively the coupling constant should take a critical value (Wilson, 1974).

Different formulations of discrete space-time were based on the introduction of momentum space of constant curvature (Snyder, 1947, 1948; Gelfand, 1959; Kadyshevsky, 1961; Kogut et al., 1976). Such approaches introduced discrete noncommutative coordinates and modified invariance properties at short distances.

The phenomenological success of the lattice approach (Kogut et al., 1976; Kogut, 1976), as well as the attractivity of the ideas prompted us to search for whether the concept of discrete space–time might support a more fundamental treatment. This has led us to basically new concepts.

In our model (Dadić and Pisk, 1977), the space structure is discrete but not frozen as in the lattice approach; on the contrary, it is "amorphous" and changes with time. We represent the space as a set of abstract objects with certain relations of neighborhood among them. The structure consisting of objects and neighborhood relations is recognized as a graph [in terms of the mathematical theory of graphs (Essam and Fischer, 1970; Harary, 1969)]. Accordingly, the lattice is only a special kind of graph.

The metric of the graph is given by graph theory. The graph does not possess dimension in a traditional sense, but our intuitive feeling can be translated into an appropriate mathematical expression (Kraemer et al., 1974). The dimension thus defined is scale dependent. It is generated through the model dynamics and thus becomes a dynamical variable.

As the dynamics should not depend on point and line labels, an appropriate symmetrization is necessary. It comes out that the standard creation and annihilation operator approach leads to contradiction when applied to graphs; instead, we define a reformulated Fock-space method based on a topological description.

We define and solve a simple model in the framework of "local" dynamics. The model suggests that only relative motion has a definite meaning. In addition, the spectrum of the model is relativistic in a certain sense.

The present paper is divided into six sections. In Section 2 we introduce our approach in an intutive way; we define the metric and the notion of dimension. In Section 3 we build a Hilbert space of states and reformulate the Fock-space method. Section 4 is devoted to "local" dynamics and a solvable example. Section 5 contains our physical picture.

2. GRAPH AS A METRIC DISCRETE SPACE; NOTION OF DIMENSION

If we try to analyze what is essential in our intuitive notion of space arising from experience as well as science, we find that it is the existence of some objects and the relation among them. This relation may be identified as a relation of neighborhood, and the way the objects interact is closely related to it.

Now we use this extraction to build our concept of discrete-space structure. Suppose there is a finite or infinite set of objects $S=\{a,b,\dots\}$. These objects are abstract in all their properties except for a relation of neighborhood. To each object we adjoin a set of objects $a_1,a_2,\dots$, which we call neighbors. The neighborhood relation n is a relation with the following properties:

1. It is a relation between two objects: anb.
2. It is commutative: $anb\langle===\rangle bna$.
3. It is not identical: $a\not n a$.
4. It is not transitive: $anb,\ bnc\langle=\neq=\rangle anc$.
5. It is exclusive: there is no sense in discussing double neighborhood between two objects.

The structure consisting of all objects and neighborhood relations with properties 1–5 is well known in mathematics (Essam and Fischer, 1970; Harary, 1969); it is called the graph and has been the subject of extensive investigations. In order to see this identity, we introduce a one-to-one correspondence between objects and points (vertices) of a graph and between neighborhood relations and lines (edges). This correspondence is shown in Figure 1.

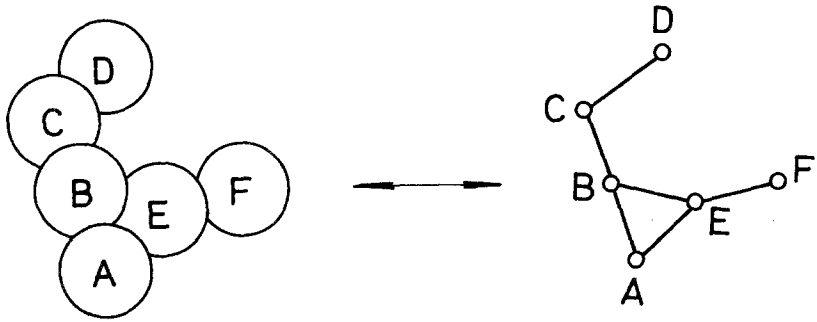

Fig. 1. Correspondence: (object, neighborhood)→(point, line).

Properties 2, 3, and 5 are restrictions on general graphs in the sense that our graphs are undirected, do not possess loops, and are not multigraphs. There are additional restrictions of statistical and dynamical origin: graphs should be unlabeled and connected.

A graph with such properties is a possible realization of discrete-space structure.

The distance $d(a,b)$ between two points a and b on a graph is naturally defined (Essam and Fischer, 1970) as a number of lines n_l in the shortest path with the initial point a and the terminal point b:

$$d(a,b) = \min_{\text{path}} n_l(a,b,\text{path}) \tag{2.1}$$

The distance function defined by (2.1) is positive definite and satisfies the following conditions:

1. $d(a,b) \geqslant 0$
2. $d(a,b) = d(b,a)$
3. $d(a,b) + (d(b,c) \geqslant d(a,c)$
4. $d(a,b) = 0 \langle = = = \rangle a = b$ (2.2)

By convention, $d(a,b) = \infty$ if there is no path relating a with b, i.e., a and b belong to the disconnected components of the graph. As the dynamics is local, the disconnected components remain disconnected and this restricts our further considerations to connected graphs only. In defining (2.1) we took freedom in the choice of the unit of length: it is such that the shortest distance (l_1, the length of the line) is 1 in these units:

$$d_{\min}(a,b) = l_1 = 1, \qquad a \neq b \tag{2.3}$$

Having the notion of metric, we are able to consider the question of dimension. It is clear that the mathematical definition of dimension valid for a continuous space is not applicable to a space with discrete metric because of the limiting procedure. The usefulness of this definition as a procedure for determining the dimension of some object experimentally, is doubtful even for a continuous space. The reason is that there is no probing particle with infinitely small wavelength. On the contrary, the range of wavelength is limited and therefore we have to raise the question of dimension for each range of energies separately. Our definition will be based on the definition of the physical dimension of some object given by Kraemer, Nielsen, and Tze (1974):

$$D(\rho) = -\frac{d \ln m(\rho)}{d \ln \rho} \tag{2.4}$$

where $m(\rho)$ is the minimal number of spheres S_r of radius $r \leqslant \rho$ needed to cover the object. When $\rho \to 0$, $D(0)$ is just the mathematical dimension.

Now we generalize the above definition to the case of a graph. Consider the region M of a graph defined by

$$M = \{a; d(a,b) \leqslant K\} \tag{2.5}$$

a and b are points on the graph, $d(a,b)$ is the distance function given by (2.1), and K is an integer. In the same way we define a sphere of radius ρ around some point c,

$$S(c,\rho) = \{a; d(a,c) \leqslant \rho\} \tag{2.6}$$

Analogously to (2.4), we define $m(\rho)$ as the minimal number of spheres $S(c,\rho)$ necessary to cover the region M. Now the ρ dimension of M is the following (differentials are going into differences):

$$D_M(\rho) = -\frac{\ln m(\rho+1) - \ln m(\rho)}{\ln(\rho+1) - \ln \rho} \tag{2.7}$$

This definition will be further modified by quantum mechanics (Kraemer et al., 1974) by using the quantum mechanical averages $\langle m(\rho) \rangle$ instead of $m(\rho)$. For the macroscopic region on a cubic lattice, this definition gives $D_M(\rho) = 3$ provided $1 \ll \rho \ll K$. It is evident that a graph may have regions of different dimensions. We expect that the requirement of dynamical stability will be strong enough to force the system to choose $D(\rho)$ fixed in almost all regions for a scale ρ large enough.

3. A REFORMULATED FOCK-SPACE METHOD

Now we have to construct a Hilbert space of states keeping in mind that the points and lines are not distinguishable. For pedagogical reasons as well for clarification of different aspects of the problem, we construct $\mathcal{H}$ in several steps.

In the first step, a vector in $\mathcal{H}$ is adjoined to each connected graph with labeled points and lines. Vectors adjoined to different graphs are orthogonal. A set of all such vectors forms a basis in $\mathcal{H}$.

$\mathcal{H}$ constructed in such a way is too large. Keeping the point labels fixed and choosing a proper symmetrization of lines, we construct a vector corresponding to a graph with labeled points and unlabeled lines. The fact that no more than one line can relate two points leads to the choice between Fermi and Pauli[1] statistics. According to the chosen statistics, we introduce creation and annihilation operators for lines with commutation relations: for Fermi statistics,

[1]Pauli statistics is widely used in the theory of ferromagnetism for a correct treatment of spin waves. See, for example, Dyson, (1956).

$$\{\beta_{ij}, \beta_{kl}^{\dagger}\} = \delta_{ik}\delta_{jl} \tag{3.1}$$

for Pauli statistics,

$$|\beta_{ij}, \beta_{kl}^{\dagger}| = 0 \qquad (i,j) \neq (k,l) \tag{3.2}$$

$$\{\beta_{ij}, \beta_{ij}^{\dagger}\} = 1 \tag{3.3}$$

The indices i and j $(i<j)$ label the end points of a given line. The vacuum state to which the operators β_{ij} apply corresponds to a "graph" consisting of labeled points only.

Thus we are led to two subspaces of $\mathcal{H}$: $\mathcal{H}_{\mathrm{AL}}$ with antisymmetrized lines and labeled points, and $\mathcal{H}_{\mathrm{SL}}$ with symmetrized lines and labeled points.

In the third step, we perform symmetrization over points. The single state is now represented by an unlabeled graph (unlabeled points, unlabeled lines). According to the type of symmetrization, such states form a basis in one of the four subspaces $\mathcal{H}_{\mathrm{SL,SP}}$, $\mathcal{H}_{\mathrm{SL,AP}}$, $\mathcal{H}_{\mathrm{AL,SP}}$, and $\mathcal{H}_{\mathrm{AL,AP}}$.

After symmetrization, the line production operator should refer not to labels (which are now absent), but to points identified by the topological description on a given graph. However, application of this operator changes the topological description of all points and lines, so that the multiplication of two or more operators referring to the topology of the same graph does not make sense. In other words, when labels are absent there is no unique way to identify (topologically) the "same" point on different graphs.

As a consequence, a simple Fock-space method, expressed through creation and annihilation operators with commutation relations such as (3.1) or (3.2) and (3.3), cannot be used for construction of the basis in any of the spaces $\mathcal{H}_{\mathrm{SL,SP}}$, $\mathcal{H}_{\mathrm{SL,AP}}$, $\mathcal{H}_{\mathrm{AL,SP}}$, $\mathcal{H}_{\mathrm{AL,AP}}$.

In order to *reformulate* the Fock-space method, we consider two (unlabeled) graphs G and G', which differ in one line. The corresponding (symmetrized) states are $|G\rangle$ and $|G'\rangle$. We define the operators $b_{GG'}^{\dagger}$, $b_{GG'}$:

$$\begin{aligned} b_{GG'}^{\dagger}|G''\rangle &= \delta_{G''G}\kappa_{GG'}^{*}|G'\rangle \\ b_{GG'}|G''\rangle &= \delta_{G''G'}\kappa_{GG'}|G\rangle \end{aligned} \tag{3.4}$$

or otherwise,

$$\langle G''|b_{GG'}|G''\rangle = \delta_{GG''}\delta_{G'G''}\kappa_{GG'} \tag{3.5}$$

The operator $b_{GG'}^{\dagger}(b_{GG'})$ thus defined may be interpreted as a line creation (annihilation) operator (the line by which G and G' differ).

On the graph G we can draw an additional line leading to G', at j positions which cannot be distinguished by the topological description. In the same way, to obtain G from G', we can remove a line from l topologically equivalent positions. The numbers j and l depend on the structure of G and G': $j=j(G,G')$, $l=l(G,G')$.

We choose the constant $\kappa_{GG'}$ to depend on j and l in the following way:

$$\kappa_{GG'}=\left[(j+1)l\right]^{1/2} \tag{3.6}$$

The creation and annihilation operators thus defined are relational in the sense that they do not refer to labels ("reference frame"!) but to a definite topological structure.

Using the operators $b^{\dagger}_{GG'}$, the state G may be constructed from the vacuum state $|0\rangle$ (graph G_0 with no lines) if the chain of graphs $G_0, G_1, G_2, \ldots, G_n$, G (where the nearby graphs differ in one line) is known:

$$|G\rangle = b^{\dagger}_{G_{n-1}G_n}/\kappa_{G_{n-1}G_n}\cdots b^{\dagger}_{G_1G_2}/\kappa_{G_1G_2}b^{\dagger}_{G_0G_1}/\kappa_{G_0G_1}|0\rangle \tag{3.7}$$

Unfortunately, the chain $G_0,\ldots,G_n$, G is not unique, and different representations of $|G\rangle$ are possible.

4. THE LOCAL INTERACTION IN THE QUANTUM MECHANICS OF GRAPHS

The dynamics of a graph represents the dynamics of its structure. The change of the graph structure should be local in terms of the metric defined by (2.2). (This metric is generalized to unlabeled graphs, so that instead of to labels we refer to the topological description of the point position.)

The interaction is local in the sense that (a) the interaction involves only processes with a minimal change of a distance structure, and (b) the interaction depends only on the local properties of the graph structure.

The interaction with such properties generally includes two terms. The first term, which we call the "potential," is diagonal in the basis of graph states and is a function of graph states:

$$V=\sum_a j_a C(m_a)+\frac{1}{2}\sum_{\substack{a,b\\ d(a,b)=1}} j_{ab}D(m_a,m_b)+\cdots \tag{4.1}$$

Here j_a is the number of equivalent points of class a, j_{ab} is the number of equivalent lines defined by the points belonging to classes a and b, m_a and

m_b are the number of lines terminal to the points from classes a and b, respectively, and $C(m_a)$ and $D(m_a,m_b)$ are some functions.

The second term, which we call "kinetic energy," is graphically represented in Figure 2. Only those elements of the graph structure are shown which are essential for the interaction. The interpretation of Figure 2 is evident. Diagram (a) represents the creation (annihilation) of one line (only when two points have a common neighboring point), diagram (b) represents the creation of one point and one line, and diagram (c) represents a jump of one line.

Obviously, the "potential" and "kinetic energy" terms do not commute, and thus a stationary state will be a linear combination of graph states with different configurations.

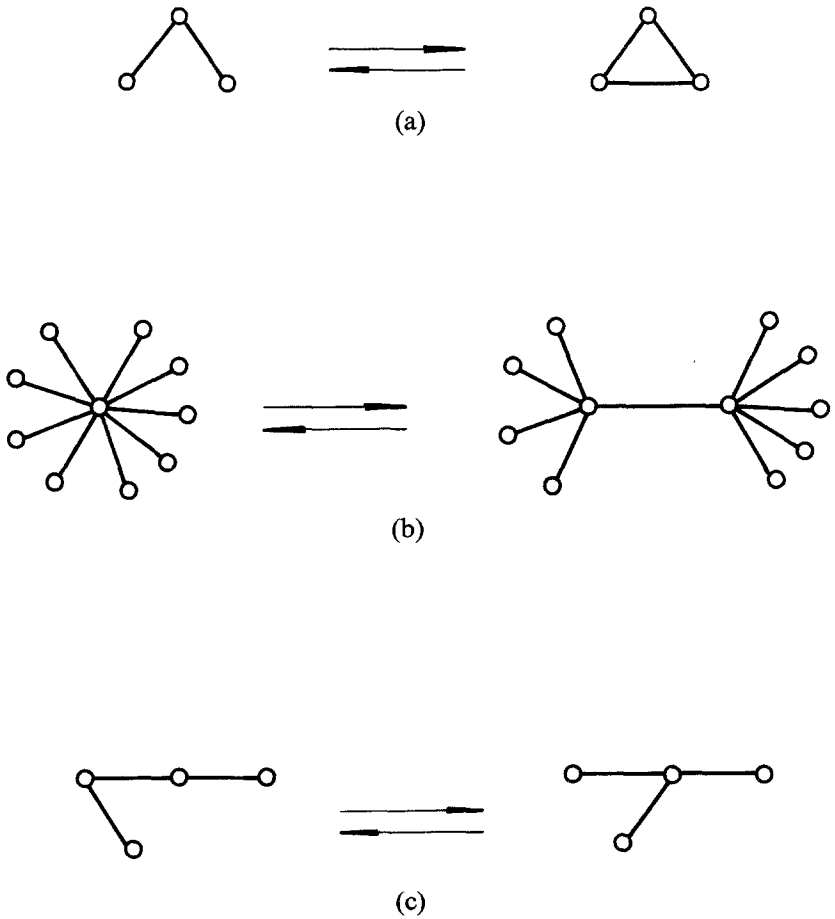

Fig. 2. Graphical representation of "kinetic energy."

The above scheme is the Hamiltonian formulation of the quantum mechanics of graphs. In this formulation, distances are discrete but time is continuous. The question whether time may or must also be discrete remains open.

To gain an insight into graph dynamics, we have to search for a solvable example. The prize for solvability will be in weaker locality constraint.

We define the "potential energy" in the following way:

$$V = \begin{cases} Y(N_t - N_r) & \text{"allowed" graph} \\ 0 & \text{"forbidden" graph} \end{cases} \tag{4.2}$$

Here N_t is the number of triangle subgraphs, N_r is the number of rhomb subgraphs, and Y is a constant ($Y = Y^*$).

In this example we define the graph as "allowed" if it is a simple chain graph which in addition contains only rhombs and triangles as subgraphs (Figure 3). The graph is "forbidden" if it is not "allowed."

The "kinetic energy" is defined as

$$T = \frac{Z}{2} \sum_{G,G'}{}' (b^\dagger_{GG'} + b_{G'G}) \tag{4.3}$$

The summation runs only over allowed graphs; Z is a constant ($Z = Z^*$).

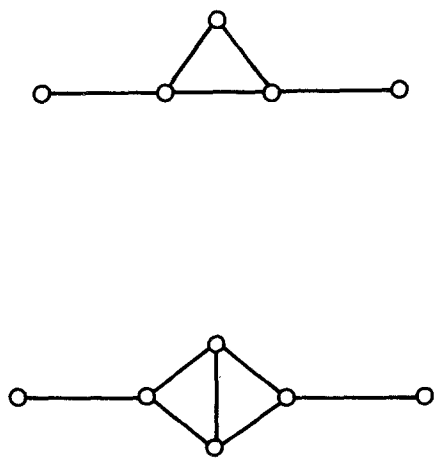

Fig. 3. Triangle and rhomb subgraph.

The "kinetic energy" thus defined transforms triangle subgraphs into rhomb subgraphs and vice versa. Because of that, the constant of motion is the sum

$$N = N_t + N_r \tag{4.4}$$

To perform some calculation, we introduce the following notation for the "allowed"-graph states:

$$|G\rangle = |s_1, s_2, \ldots, s_n; N; d_1, d_2, \ldots, d_{N-1}\rangle \tag{4.5}$$

Here for the ith subgraph from the left, $s_i = (+1$ for triangle, -1 for rhomb) subgraphs. In this example we distinguish the left from the right side of an infinite chain graph, but this is not essential for our main conclusions. $d_1, \ldots, d_{N-1}$ are relative distances between neighboring subgraphs.

Now we can perform calculations inside the invariant subspaces of a given N.

For $N=0$ there is a single eigenstate $|0\rangle$ corresponding to a simple infinite chain:

$$H|0\rangle = (T+V)|0\rangle = 0$$

For $N=1$ we have

$$H|+1;1\rangle = Y|+1;1\rangle + Z|-1;1\rangle$$

$$H|-1;1\rangle = Z|+1;1\rangle - Y|-1;1\rangle$$

The solutions are as follows:

$$|\pm\rangle = \left[Z^2 + (Z \mp Y)^2\right]^{-1/2} \left[Z|+1;1\rangle + (\pm Z - Y)|-1;1\rangle\right]$$

$$H|\pm 1\rangle = \pm\left[Y^2 + Z^2\right]^{1/2} |\pm 1\rangle$$

For $N=2$ we have

$$H|1,1;2;d\rangle = Y|1,1;2;d\rangle + \tfrac{3}{2}\big[|1,-1;2;d\rangle$$

$$+ |-1,1;2;d\rangle + (1-\delta_{d1})(|1,-1;2;d-1\rangle + |-1,1;2;d-1\rangle)\big]$$

The solutions are classified with the help of the "relative momentum" p, and the eigenvalues are defined as

$$\begin{aligned} h_{\lambda_1\lambda_2}(p) &= \lambda_1 h(p) + \lambda_2 h(-p) \\ \lambda_i &= \pm 1 \\ h(p) &= \left[Y^2 + 4Z^2 \sin^2 \frac{p}{2} \right]^{1/2} \end{aligned} \tag{4.6}$$

For general N, the momenta $p_1, \ldots, p_N$ are introduced with the condition

$$\sum_{i=1}^{N} p_i = \left[1 + (-1)^N \right] \frac{\pi}{2} \tag{4.7}$$

and the eigenvalues are

$$h_{\lambda_1 \cdots \lambda_N}(p_1 \cdots p_N) = \sum_{i=1}^{N} \lambda_i h(p_i) \tag{4.8}$$

The properties of the model are significant:

(a) The model describes the dynamics of N "particles" in a one-dimensional discrete world. The "particles" are almost free with the short-range interaction only.

(b) It is not possible to introduce absolute motion. In graph dynamics, the natural reference frame appears as the unique possibility.

(c) The spectrum is relativistic in the following sense: The spectrum is symmetric with respect to zero. In the expression for single-particle energy (4.6), the correspondence $c = Z$; $m = Y/Z^2$ leads to the expression

$$h(p) = \left[m^2 c^4 + 4c^2 \sin^2 \frac{p}{2} \right]^{1/2} \tag{4.9}$$

which is relativistic in the region $|p|/2 \ll 1$. Under the condition $Y/Z \ll 1$, this behavior extends to the regions where the group velocity differs from c by an amount that is arbitrarily small.

5. THE WORLD OF PHYSICS IN THIS FRAMEWORK

Most of the physical insights given in the preceding sections depend on the special choice of the Hamiltonian. We can discuss some properties in the framework of our simple solvable model, or by assuming the existence of a Hamiltonian which possesses a richer structure. So we may raise the question how to translate the known phenomena into the language of such a theory.

The phenomena we want to discuss here are vacuum, particles, and relativity. We shall try to stress the differences between our approach and continuous-time theories.

Vacuum must be three dimensional and isotropic, at least in the whole range where quantum electrodynamics (QED) is tested to be valid. This leads us to the conclusion that the ground state of the system should have a dimension $D(\rho)=3$ for some $\rho \rangg l_2$, and l_2 should be smaller than the lower limit of distance of validity of QED. l_2 and $D(\rho)$ are quantities calculable from theory. Thus we immediately obtain two scales: l_1 is the distance between two points (1 in the chosen units) and l_2 is the scale at which the ground state exhibits a definite ρ dimension. With respect to this, vacuum in our solvable model is one dimensional for all $\rho \gg 1$ because of the simplicity of the Hamiltonian.

In our framework, particles may appear in two basically different ways. In one way, which is more attractive for us, a particle is a low-lying excited state generated by the same interaction as the ground state. This excited state differs from the ground state by local peculiarities in structure of graphs states entering a linear combination. This excited state must possess a certain degree of stability and it is evidently an extended object. This introduces new scales: radii of extension of particles.

The type of peculiarities will depend on the chosen interaction, but it is interesting to point out a possibility which is the particular property of this model and which is absent both in continuous space–time models and in lattice models. It is possible that the dimension of a particle (i.e., the ρ dimension of the region of peculiarities) is different from the ρ dimension of the ground state. There are two interesting possibilities:

1. The ρ dimension varies when approaching the center of the peculiarity region.

2. The ρ dimension of the peculiarity region is fixed but is not that of the ground state. For example, $D(\rho)$ may be 1 or 2 (strings and plates already known from different models), but it may also be equal to 4 or even larger. This offers a possibility of natural appearance of additional symmetries (broken and exact).

Our solvable model exhibits these features only in part. We can identify a linear combination of rhombs and triangles as particles, but these peculiarities are essentially one dimensional as a consequence of a dynamics which is too simple.

The other way of introducing particles is to put them in by hand through new degrees of freedom, i.e., additional "charged" points and lines which carry the quantum numbers of particles. Such an approach is close to the way quarks are introduced into Wilson's lattice.

In our approach, the problem of fixing the frame is more serious than in continuous-space or lattice theories. This is because graph states, which

are a basis of the Fock space, are symmetrized, and only the topological structure of graphs is relevant. As a consequence, only relative distances between particles (peculiarities of the structure) may be meaningful. Our solvable model proves these features. Even more, the relativistic spectrum of this model is compatible with a natural frame in which the total momentum is zero. The picture of space emerging in this way consists of particles and ground-state-like structures spanned between them. Thus, observables (distances, momentum,...) can be defined only for many-particle states. For a more realistic Hamiltonian, this definition has evident limitations corresponding to the scale l_2 and the radii of particles.

The dynamics of the model is local and thus the speed of a signal cannot be infinite, and we may expect some kind of relativistic invariance. Spontaneous breakdown of the symmetry is, of course, possible; for example, in the case of nonlocal interaction or if the degrees of freedom are drastically restricted.

Keeping in mind the above discussions, we may conclude that invariance will be more "general-relativity"-like and this makes it plausible to settle the fundamental length of gravity somewhere in the range between l_1 and l_2.

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